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The phases of two-dimensional spin and four-dimensional gauge systems with $Z(N)$ symmetry

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Abstract. Using a generalised duality transformation, symmetry considerations and assuming criticality to be continuous in the system parameters, we obtain the phase diagram for two-dimensional $Z(N)$ spin models (four-dimensional gauge $Z(N)$ models). Besides the phases characterised by the spontaneous breakdown of $Z(N)$ symmetries for spin systems (the behaviour of the Wilson loop for gauge systems), we predict the existence of a soft phase characterised by the vanishing of all powers of order and disorder parameters (a Wilson and 't Hooft loop decaying together with all powers like the perimeter). For the spin system phases with non-vanishing order and disorder parameters are forbidden when those parameters obey non-trivial commutation relations. For gauge systems all combinations of Wilson and 't Hooft loops decaying as the area and the perimeter are allowed. Duality relations for three-dimensional gauge plus Higgs system are given.

1. Introduction

Lattice gauge theories have been intensively studied recently (Wilson 1974, Fradkin and Susskind 1978, 't Hooft 1978, Kogut 1979, Elitzur *et al* 1979, Horn *et al* 1979, Ukawa *et al* 1979, Mack 1980, Creutz *et al* 1980) because they arise naturally from Euclidean gauge theories by introducing a lattice in order to provide a short distance cut-off. One of the hurdles to be overcome by gauge theories, if they are to be taken seriously as candidates for a theory of hadrons, is the explanation of the confinement of quarks. If the symmetry of the gauge theory is taken to be $SU(N)$, then its centre $Z(N)$ seems to play a crucial role in implementing confinement ('t Hooft 1978, 1979).

If this is true, a preliminary step in the study of gauge theories with quarks and gluons would be the study of $Z(N)$ gauge theories on a four-dimensional lattice. It is furthermore instructive to look also at two-dimensional spin models with global $Z(N)$ symmetry (Fradkin and Susskind 1978, Kogut 1979, Domany and Riedel 1979, Balian *et al* 1975, Bellisard 1978, Korthals Altes 1978, Alcaraz and Köberle 1980), because of the many ways in which they are analogous to four-dimensional gauge models and because of their greater simplicity, which allows one to obtain exact results (Baxter 1973, Hintermann *et al* 1978, Köberle and Swieca 1979, Köberle and Kurak 1980).

In this paper we study the phase diagram of two-dimensional $Z(N)$ spin and four-dimensional $Z(N)$ gauge systems (without matter fields). We use two main tools

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for this purpose. One is a kind of self-duality exhibited by these systems and the second is a continuity hypothesis: we assume that criticality is continuous in the system parameters. Whereas for spin systems this seems to be a rather plausible assumption, for gauge systems it really is a strong one, since they exhibit first-order transitions without a local order parameter.

For spin systems we obtain phases which can be characterised by the behaviour of powers of order and disorder parameters $\langle S^n \rangle$ and $\langle \rho^n \rangle$. For $N>4$ we predict the existence of a soft (massless) phase with $\langle S^n \rangle = \langle \rho^n \rangle = 0$, $n \neq 0, N$. A short account of these results was given in Alcaraz and Köberle 1980†. We correct here an erroneous statement made there: $N=4$ does not possess a soft phase as claimed in AK, but this soft phase appears only for $N>4$.

For gauge systems our proposed phase diagrams look exactly as in the spin case, but the order of the phase transitions changes and we have only non-local order and disorder parameters: the Wilson loop and its powers $\langle A^n(C) \rangle$ and 't Hooft's disorder loop and its powers $\langle B^n(\tilde{C}) \rangle$. Here C and $\tilde{C}$ are closed curves on the direct and dual lattice. There is a rough correspondence

$$\langle S^n \rangle \neq 0 \leftrightarrow \langle A^n(C) \rangle \approx \exp(-P)$$

$$\langle S^n \rangle = 0 \leftrightarrow \langle A^n(C) \rangle \approx \exp(-A)$$

$$\langle \rho^n \rangle \neq 0 \leftrightarrow \langle B^n(\tilde{C}) \rangle \approx \exp(-P)$$

$$\langle \rho^n \rangle = 0 \leftrightarrow \langle B^n(\tilde{C}) \rangle \approx \exp(-A)$$

where A and P are the enclosed area and the perimeter of the curves C and $\tilde{C}$, although this translation scheme is incorrect in important details. For example, we may have phases in which both $\langle A^n(C) \rangle$ and $\langle B^n(\tilde{C}) \rangle$ have a perimeter decay, but if S and ρ have non-trivial commutation relations, they cannot simultaneously have non-vanishing averages. We again predict a soft phase for $N>4$ in which all powers (except $n=0, N$) of $\langle A^n(C) \rangle$ and $\langle B^n(\tilde{C}) \rangle$ decay as the perimeter of the curves C and $\tilde{C}$.

In § 2 we set up the transfer matrix and duality formalism for spin systems, which carries over identically for gauge systems. In § 3 we explain our results for spin systems considering $N<8$ for simplicity of presentation. $N=6$ is an interesting example which is worked out in some detail. Gauge systems are treated in § 4, where all four types of phases predicted by t' Hooft (1978) are shown to be realised. In § 5 we briefly sum up our conclusions.

Mean-field calculations in support of our claims are presented in appendix 1. For three dimensions the $Z(N)$ gauge plus Higgs system is self-dual and it is briefly introduced in appendix 2.

2. Transfer matrix and duality for spin systems

Our $Z(N)$ spin system is defined on a two-dimensional square lattice with sites labelled by the vector $\boldsymbol{i}$. On each lattice site there is a spin variable $S(\boldsymbol{i})$ satisfying

$$[S(\boldsymbol{i})]^N = 1. \tag{2.1}$$

† Henceforth referred to as AK.

We may thus equivalently introduce integer-valued variables $n(\boldsymbol{i})$ for each site:

$$S(\boldsymbol{i}) = \exp[(2\pi \mathrm{i}/N)n(\boldsymbol{i})], \qquad n(\boldsymbol{i}) = 0, 1, 2, \ldots, N-1. \tag{2.2}$$

If we restrict ourselves to nearest-neighbour couplings of ferromagnetic character, the most general $Z(N)$-invariant action will be

$$A = -\sum_{\langle ij\rangle}\Big\{\frac{J_1}{2}(S^+(\boldsymbol{i})S(\boldsymbol{j}) + \mathrm{cc} - 2) + \frac{J_2}{2}[(S^+(\boldsymbol{i})S(\boldsymbol{j}))^2 + \mathrm{cc} - 2] + \ldots + \frac{J_{\bar N}}{2}[(S^+(\boldsymbol{i})S(\boldsymbol{j}))^{\bar N} + \mathrm{cc} - 2]\Big\} \tag{2.3a}$$

$$= -\sum_{\langle ij\rangle}\Big(J_1\Big\{\cos\Big[\frac{2\pi}{N}(n(\boldsymbol{i}) - n(\boldsymbol{j})) - 1\Big\} + J_2\Big\{\cos\Big[\frac{4\pi}{N}(n(\boldsymbol{i}) - n(\boldsymbol{j}))\Big] - 1\Big\} + \ldots + J_{\bar N}\Big\{\cos\Big[\frac{2\pi \bar N}{N}(n(\boldsymbol{i}) - n(\boldsymbol{j}))\Big] - 1\Big\}\Big) \tag{2.3b}$$

where $\langle \boldsymbol{i}, \boldsymbol{j}\rangle$ indicates a sum over nearest-neighbour sites and $\bar N$ is the largest integer smaller than or equal to $N/2$.

Our discussion will be based on the transfer matrix T. It is an operator acting on the Hilbert space H as follows. If we call the vertical direction of our two-dimensional lattice the 'time' axis, then at a particular time $t = t_0$, the state of the system is described by a vector in the direct-product Hilbert space H spanned by $|n\rangle = \Pi_{i=-M/2}^{+M/2}|n(i)\rangle$, where $|n(i)\rangle$ describes the spin state at site i at $t = t_0$ and M is the size of the lattice. H thus has dimension $(N)^M$. We choose $|n(i)\rangle$ as eigenstates of the unitary operator $S(i)$:

$$S(i)|n(i)\rangle = \exp[(2\pi \mathrm{i}/N)n(i)]|n(i)\rangle. \tag{2.4}$$

We define T as the operator whose matrix elements are $\exp(-\beta A)$:

$$\langle n'|T|n\rangle = \exp\Big\{\sum_{\langle ij\rangle}\sum_{\alpha_i=1}^{\bar N}\frac{K_{\alpha_i}}{2}[(S^+(i)S(j))^{\alpha_i} + \mathrm{cc} - 2]\Big\} \tag{2.5}$$

where $|n\rangle$ and $|n'\rangle$ are the state vectors of two nearest neighbouring rows. T consists of two factors: $T = T_1T_2$, where T_1 contains the interactions within a row and is thus diagonal:

$$T_1 = \exp\sum_i\sum_{\alpha_i=1}^{\bar N}\frac{K_{\alpha_i}}{2}[(S^+(i)S(i+1))^{\alpha_i} + \mathrm{HC} - 2]. \tag{2.6}$$

T_2 contains the off-diagonal terms of the form

$$\exp\Big\{\sum_{\alpha_i}\frac{K_{\alpha_i}}{2}[(S^+(\boldsymbol{i})S(\boldsymbol{j}))^{\alpha_i} + \mathrm{cc} - 2]\Big\} \tag{2.7}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are nearest-neighbour sites on adjacent rows. In order to express (2.7) as a matrix element of some operator, we introduce rotation operators $R(i)$, which rotate the spin on site i by an angle $2\pi/N$:

$$R(i)|n(i)\rangle = |n(i) + 1\rangle (\mathrm{mod}\ N). \tag{2.8}$$

Our basic operators $S(i)$ and $R(i)$ obey the relations

$$S^N = R^N = 1, \qquad \mathrm{Tr}(S) = \mathrm{Tr}(R) = 0$$
$$R^\alpha R^\beta = R^{\alpha+\beta} (\bmod N) \tag{2.9}$$

$RS = \exp(2\pi \mathrm{i}/N)SR$ (for S and R on the same site, commuting otherwise).

If we expand the exponential in (2.7) using $S^N = 1$ the result may be written in terms of the matrix elements

$$\langle n'(i)|R^\alpha|n(i)\rangle = \delta_{n',n+\alpha} (\bmod N) \tag{2.10}$$

as

$$T_2 = \prod_i \sum_{\eta=0}^{N-1} [R(i)]^\eta \exp\left\{\sum_{\alpha=1}^{\tilde{N}} K_\alpha\left[\cos\left(\frac{2\pi\alpha\eta}{N}\right) - 1\right]\right\}. \tag{2.11}$$

Using the identity†

$$\sum_{\eta=0}^{N-1} R^\eta \exp\left\{\sum_{\alpha=1}^{\tilde{N}} K_\alpha\left[\cos\left(\frac{2\pi\alpha\eta}{N}\right) - 1\right]\right\} = \exp\left\{\sum_{\alpha=0}^{\tilde{N}} \frac{f_\alpha(K)}{2}(R^\alpha + \mathrm{HC})\right\} \tag{2.12}$$

based on $R^N = 1$, we finally obtain T as

$$T = \exp \sum_i \sum_{\alpha_i=1}^{\tilde{N}} \frac{K_{\alpha_i}}{2}\{[S^+(i)S(i+1)]^{\alpha_i} + \mathrm{HC} - 2\} \exp \sum_j \sum_{\alpha_j=0}^{\tilde{N}} \frac{f_{\alpha_j}(K)}{2}\{[R(j)]^{\alpha_j} + \mathrm{HC}\}. \tag{2.13}$$

Summing over a complete set of intermediate states, we see that the partition function can be expressed as

$$Z(K) = \mathop{\mathrm{Tr}}_{\{S(i)\}} [T]^M, \qquad M \to \infty \tag{2.14}$$

where we choose periodic boundary conditions.

We now define dual variables (Kadanoff and Ceva 1971) as follows:

$$\sigma(\tilde{i}) = S(i)S^+(i+1)$$
$$\rho(\tilde{i}) = \prod_{j<i} R^+(j)R^+(i) \tag{2.15}$$

where the non-local disorder variable $\rho(\tilde{i})$ satisfies

$$\rho^+(\tilde{i})\rho(\tilde{i}+\tilde{1}) = R^+(i+1). \tag{2.16}$$

With the aid of the variables (2.15) our transfer matrix may be expressed as

$$T = \exp \sum_i \sum_{\alpha_i=0}^{\tilde{N}} \frac{f_{\alpha_i}(K)}{2}\{[\rho^+(\tilde{i})\rho(\tilde{i}+\tilde{1})]^{\alpha_i} + \mathrm{HC}\} \exp \sum_j \sum_{\alpha_j=1}^{\tilde{N}} \frac{K_{\alpha_j}}{2}\{[\sigma(\tilde{j})]^{\alpha_j} + \mathrm{HC} - 2\}. \tag{2.17}$$

Since $\sigma(\tilde{i})$ and $\rho(\tilde{i})$ satisfy the same $Z(N)$ algebra as $S(i)$ and $R(i)$, we see from equation (2.17) that our system is self-dual in the sense that

$$Z(\{K_\alpha\}, \{f_\beta\}) = eZ(\{f_\alpha\}, \{K_\beta\}) \tag{2.18}$$

where e is an irrelevant constant.

† The functions f_α defined here differ by a factor of two from those of AK.

Our duality transformation $\mathscr{D}: K_\alpha \leftrightarrow f_\alpha$ may be more conveniently expressed in terms of the variables

$$x_\alpha \equiv \exp\left\{\sum_{\delta=0}^{N-1} K_\delta\left[\cos\left(\frac{2\pi\delta\alpha}{N}\right)-1\right]\right\}, \qquad x_{N-\alpha} = x_\alpha (\bmod N). \tag{2.19}$$

Diagonalising the cyclic matrix R (Wu and Wang 1976) we may immediately solve equation (2.12) for the function f_α in order to obtain an expression for the dual variables $\tilde{x}_\alpha = \mathscr{D}(x_\beta)$:

$$\begin{aligned}\tilde{x}_\alpha &\equiv \exp\left\{\sum_{\delta=0}^{N-1} f_\delta\left[\cos\left(\frac{2\pi\delta\alpha}{N}\right)-1\right]\right\}\\ &= \left\{\sum_{\delta=0}^{N-1} \exp(2\pi \mathrm{i}\delta\alpha/N)x_\delta\right\}\Big/\left\{\sum_{\delta=0}^{N-1} x_\delta\right\}.\end{aligned} \tag{2.20}$$

The transformation $x_\alpha \to \tilde{x}_\alpha$ maps one $Z(N)$ model with coupling constants K_α into another $Z(N)$ model with coupling constants $f_\alpha(K)$. Yet there are some models which are self-dual (in the sense of Kramers and Wannier 1941). For these special models the transformation $K_\alpha \leftrightarrow f_\alpha(K)$ is equivalent to a change in temperature. This occurs for the Potts model (Potts 1952), where $K_1 = K_2 = \ldots = K_N$ and $f_1 = f_2 = \ldots = f_N$ so that the temperature change $kT \to kT' = J_\alpha/f_\alpha$ implements $x_\alpha \to \tilde{x}_\alpha$. Another model, albeit with temperature-dependent couplings J_α, is the Villain model (Villain 1975). It approximates the model defined by $K_\alpha = K\delta_{\alpha 0}$ at low and high temperatures and its partition function is given by

$$Z_{\mathrm{V}} = \sum_{\{n_i\}} \exp\left\{\sum_{\langle ij\rangle} g_\beta\left[\frac{2\pi}{N}(n_i - n_j)\right]\right\} \tag{2.21}$$

where

$$\exp g_\beta(x) = \sum_{n=-\infty}^{+\infty} \exp\left[-\frac{\beta}{2}(x-2\pi n)^2\right]. \tag{2.22}$$

The Villain coupling constants x_α^{V} are

$$x_\alpha^{\mathrm{V}}(T) = \frac{\sum_{n=-\infty}^{+\infty} \exp[-(2\pi^2/N^2T)(\alpha - Nn)^2]}{\sum_{n=-\infty}^{+\infty} \exp(-2\pi^2 n^2/T)} \tag{2.23}$$

and their transformed couplings $\tilde{x}_\alpha^{\mathrm{V}}(T)$ are given by

$$\tilde{x}_\alpha^{\mathrm{V}}(T) = x_\alpha^{\mathrm{V}}(4\pi^2/N^2T). \tag{2.24}$$

From equation (2.20) we see that the hyperplane

$$\sum_{\alpha=0}^{N-1} x_\alpha = N^{1/2} \tag{2.25}$$

is left invariant by $x_\alpha \to \tilde{x}_\alpha$. This hyperplane contains the critical points of the Potts and Villain models and on it $\mathscr{D}$ is a linear transformation. Thus for $N \leq 7$, when we have only three or less independent coupling constants, the fixed points $\bar{\Sigma}$ of $x_\alpha \to \mathscr{D}(x_\alpha)$ lie on a straight line passing through the Potts and Villain critical points.

For $N > 7$ $\bar{\Sigma}$ is an $(\overline{N/4})$-dimensional hypersurface obtained by solving the first $\overline{N/4}$ equations $\tilde{x}_\alpha = x_\alpha$, $\alpha = 1, 2, \ldots, \overline{N/4}$.

3. Phase diagram for spin systems in two dimensions

Our aim is to obtain as much information as possible about the surface Σ of critical points. It is known (Fröhlich and Lieb 1979) that each of our $Z(N)$ models has at least one critical point at low enough temperature, where spontaneous magnetisation sets in, that is, for a given set of couplings J_α, the thermodynamic path Γ obtained by varying the temperature from 0 to ∞ strikes Σ at least once and at the low-temperature side we have

$$\langle S\rangle \neq 0. \tag{3.1}$$

If there is only one phase transition, then this critical point must be a fixed point of $x_\alpha \to \tilde{x}_\alpha$, that is it must satisfy

$$f_\alpha(K) = K_\alpha, \tag{3.2}$$

and since the duality transformation interchanges order and disorder we have

$$\langle\rho\rangle \neq 0 \tag{3.3}$$

at the high-temperature side. Notice that in general the duality transformation does not map Γ onto itself, but this happens only for models which are self-dual in the Kramers–Wannier sense.

If there is more than one critical point, $\mathscr{D}$ only maps one branch of Σ into another. In this case we look at limiting models, where some K_α are either zero or infinity, whose critical surface Σ we already know, and then trace out Σ by assuming that critical points do not disappear. This continuity assumption—that criticality is continuous in the parameters of the model — can easily be proved, if the phases are distinguished by a symmetry which is spontaneously broken (Landau and Lifshitz 1958), but in the present paper we will assume that it is always true.

For $N = 2$ we obtain the well known Ising model and for $N = 3$ the three-state Potts model having one critical point and a second- (or higher-) order phase transition.

For $N > 4$ we shall draw phase diagrams in terms of the variables x_α, $\alpha = 1, \ldots, \bar{N}$. We 'get a feeling' for these variables by writing them as

$$x_\alpha = \exp(-\beta\varepsilon_\alpha), \qquad 0 \leqslant \varepsilon_\alpha < \infty \tag{3.4}$$

where ε_α is the energy required to rotate a spin through an angle $2\pi\alpha/N$. Thus in the region where $\varepsilon_{\bar{N}} > \varepsilon_{\bar{N}-1} > \ldots > \varepsilon_1$ or $x_1 > x_2 > \ldots > x_{\bar{N}}$ all spin states are energetically available and at high enough β the $Z(N)$ symmetry will be completely broken. If we lower β sufficiently to gain enough entropy there will be a transition where the $Z(N)$ symmetry is completely restored. On the other hand, in the complementary region, for every subgroup of $Z(N)$ one can find a domain in x space where the corresponding symmetry is completely broken. For $N = 6$, for example, this means that there is a phase transition where the whole $Z(6)$ symmetry is broken, besides regions where $Z(6)$ breaks down to $Z(3)$ or $Z(2)$. This follows from our continuity assumption and the fact that for every $Z(N')$, N'/N = integer, one has at least one transition, by taking suitable limits $K_\alpha \to 0$ or $K_\alpha \to \infty$. As $N \to \infty$ our model becomes U(1) invariant and all the above transitions collapse to $T = 0$.

It is known (Elitzur *et al* 1979) that the Villain model, which passes through the region $x_1 < x_2 < \ldots < x_{\bar{N}}$, has more than the two phases† characterisable by $Z(N)$ symmetry only. Thus there exists an extra phase containing the self-dual line, which

† Namely $Z(N)$ is either completely broken or completely unbroken.

implies that in this phase the duality transformation interchanges order with disorder. As we cross from the low-temperature phase into this extra phase, the order parameter goes to zero and consequently in this phase $\langle S\rangle = 0 = \langle\rho\rangle$. This in turn requires (t' Hooft 1978) this extra phase to be 'soft' with power-law decaying correlations, unless this phase is still characterised by some symmetry. This happens for $N = 4$ in the region $x_1 < x_2$, where $\langle\rho\rangle = \langle S\rangle = 0$, but we still have a residual $Z(2)$ symmetry. Thus $\langle\rho\rangle$ vanishes simply due to $Z(2)$ charge conservation (as does $\langle S\rangle$) and no zero-mass behaviour can be deduced in this case as we erroneously did in AK. A similar situation occurs for $N = 6$, as we shall discuss below.

Although in special cases this zero-mass behaviour has been shown to hold (Elitzur *et al* 1979), a general proof that $\langle S^n\rangle = \langle\rho^k\rangle = 0$ with $S^n\rho^k = \exp[(2\pi i/N)nk]\rho^k S^n$, $kn \neq N$, implies massless particles is unavailable. The heuristic argument, to be made into a proof, goes as follows. A phase having $\langle S^n\rangle = 0$ is disordered and the system takes no account of boundary conditions. The kink operator ρ^n applied to this vacuum $|\rangle$ (to use the Hamiltonian language of Kogut and Susskind (1975)) usually has non-zero overlap with the original vacuum state, implying $\langle\rho^n\rangle \neq 0$. To avoid this conclusion enough long-range correlations must exist in the system so that $\rho^k|\rangle$ is orthogonal to $|\rangle$, while still maintaining $\langle S^n\rangle = 0$. The only way out is to assume a power-law decay for the correlations, which implies massless particles. Note that this argument assumes that the ρ^k and S^n operators 'know' about each other, meaning $[\rho^k, S^n] \neq 0$. For this reason the case $kn = N$ must be excluded.

Due to this extra phase a bifurcation must occur at some $[\overline{N/4}]$-dimensional hypersurface E. Since the Potts model has only one critical point for $N > 4$ (Hintermann *et al* 1978) and since the transition is of first order (Baxter 1973), the Potts critical point cannot belong to E. Otherwise we would have to join a soft phase with long-range fluctuations to a region with no such fluctuations. As we move along Σ from the Potts critical point in the direction of the critical point of the Villain model, the latent heat diminishes and we identify E with the region where the latent heat has just vanished. A mean-field calculation (appendix 1) shows just these features.

When N is a prime number everything stated above about the soft phase holds for all powers of S and ρ. For example, if $\langle S\rangle \neq 0$ at low temperature, also all powers of S must acquire non-vanishing expectations, and these will also vanish once we cross into the soft phase.

Consequently we have in this phase

$$\langle S^n\rangle = \langle\rho^n\rangle = 0, \qquad n = 1, 2, \ldots, \bar{N}. \tag{3.5}$$

For $N \neq$ prime number the symmetries of the systems do not distinguish the various powers of order and disorder parameter in the region $x_1 > x_2 > \ldots > x_{\bar{N}}$, so that we expect equation (3.5) to hold for any $N > 4$. This soft phase is bordered by two phases with $\langle S^n\rangle \neq 0$, $\langle\rho^k\rangle = 0$ and $\langle S^n\rangle = 0$, $\langle\rho^k\rangle \neq 0$, k, $n = 1, 2, \ldots, \bar{N}$ and exponentially decaying correlations. Since the soft phase is not related to the spontaneous breaking of a discrete symmetry it survives the $N \to \infty$ limit and shows up in the U(1)-invariant XY model.

Let us now analyse several instructive examples in detail.

3.1. $N = 4$

The phase diagram for this case, shown in figure 1, is already known from Wu and Lin (1974), since $Z(4)$ is a special case of the four state Ashkin–Teller (1943) model. In the

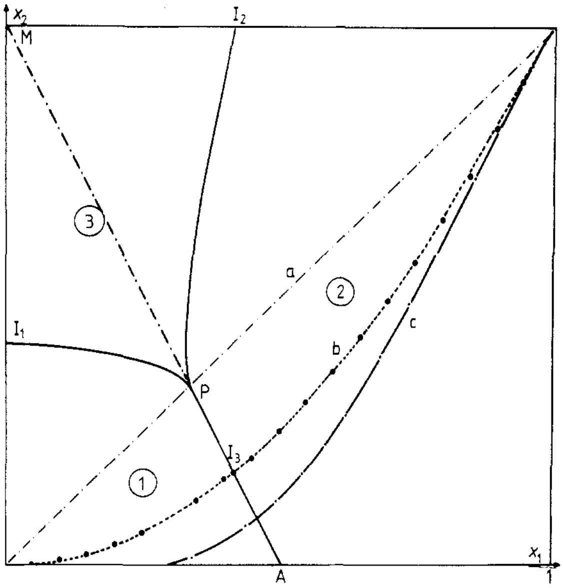

Figure 1. Schematic diagram of the $Z(4)$ model. The straight line AM is self-dual. The curves a, b and c represent the thermodynamic paths of the scalar Potts ($J_1 = J_2$), vector Potts ($J_2 = 0$) and Villain model respectively. P is the critical point of the four-state Potts model and I_1, I_2, I_3 are critical Ising points.

limit $J_2 \to \infty$, $J_1 \to 0$ or $J_2 \to 0$ we obtain Ising models with critical points I_1, I_2 and I_3 respectively.

This can most easily be seen by introducing two Ising spins σ and τ at each site, connected to S by

$$S = [\exp(i\pi/4)\sigma + \exp(-i\pi/4)\tau]/\sqrt{2}. \tag{3.6}$$

The interaction then becomes

$$A = -\sum_{\langle ij \rangle} \left[\frac{J_1}{2}(\sigma(i)\sigma(j) + \tau(i)\tau(j) - 2) + J_2(\sigma(i)\sigma(j)\tau(i)\tau(j) - 2)\right]. \tag{3.7}$$

For $J_2 = 0$ we obtain two identical decoupled Ising models (Suzuki 1967) corresponding to curve b of figure 1. For $J_1 = 0$ we obtain a system with the local gauge symmetry

$$\begin{Bmatrix} \sigma(i) \\ \tau(i) \end{Bmatrix} \to \begin{Bmatrix} -\sigma(i) \\ -\tau(i) \end{Bmatrix}. \tag{3.8}$$

The gauge transformation $\sigma \to \sigma\tau$ produces an Ising model in the variable $\sigma\tau = S^2$. At the point I_2 this variable acquires a non-zero average as we lower the temperature and the $Z(4)$ invariance is broken down to $Z(2)$.

Analogously for $J_2 = \infty$ as we decrease the temperature S acquires a non-zero average, completely breaking the $Z(4)$ invariance. Carrying out a perturbation expansion around $J_1 = 0$ and $J_2 = \infty$ we establish the following picture:

Phase 1: $\langle S \rangle \neq 0, \langle \rho \rangle = 0$; $\langle S^2 \rangle \neq 0, \langle \rho^2 \rangle = 0$; $m \neq 0$; all symmetries broken

Phase 2: $\langle S\rangle = 0, \langle\rho\rangle \neq 0$; $\langle S^2\rangle = 0, \langle\rho^2\rangle \neq 0$; $m \neq 0$; $Z(4)$ invariant

Phase 3: $\langle S\rangle = 0, \langle\rho\rangle = 0$; $\langle S^2\rangle \neq 0, \langle\rho^2\rangle \neq 0$. $\not\Rightarrow m = 0(!!)$ $Z(2)$ invariant.

Due to the $Z(2)$ invariance of phase 3 the charge selection rule operating here allows different sets of intermediate states to contribute to $\langle S^2(\boldsymbol{i})S^2(0)\rangle$ and $\langle S(\boldsymbol{i})S(0)\rangle$, which is responsible for the contrasting expectations $\langle S\rangle$ and $\langle S^2\rangle$. That this phase contains no massless excitations also follows at the $J_2\to\infty$ limit. Then S^2 is frozen, implying $\sigma=\tau$:

$$\langle S(\boldsymbol{i})S(0)\rangle|_{J_2\to\infty} = \sum_{\{\sigma\}} 2\sigma(\boldsymbol{i})\sigma(0)\exp\Big(\sum_{\langle \boldsymbol{ij}\rangle} K_1\sigma(\boldsymbol{i})\sigma(\boldsymbol{j})\Big) \tag{3.9}$$

which falls off exponentially along the line $x_1=0$ and $x_2\neq \mathrm{I}_1$, this behaviour persisting also for $x_1 = 0+\varepsilon$.

Finally notice that we have realised all combinations of $\langle S^n\rangle$ and $\langle\rho^k\rangle$ equal and not equal to zero, except $\langle S^n\rangle\neq 0$, $\langle\rho^k\rangle\neq 0$. This is impossible whenever S^n and ρ^k have non-trivial commutation rules. For, taking the average of $S^n(i)\rho^k(j) = \exp(2\pi ikn/N)\rho^k(j)S^n(i)$ and examining $(\boldsymbol{i}-\boldsymbol{j})\to\infty$, we obtain from clustering

$$\langle S^n\rangle\langle\rho^k\rangle = \exp(2\pi ink/N)\langle S^n\rangle\langle\rho^k\rangle \tag{3.10}$$

but this is impossible unless $kn = N$. In fact $\langle S^2\rangle\neq 0$ and $\langle\rho^2\rangle\neq 0$ is the only case allowed for $N=4$ and is realised in phase 3.

3.2. N = 5

We have spontaneous $Z(5)$ symmetry breaking when crossing the straight line $\overline{\mathrm{E}_1\mathrm{E}_2}$ of figure 2, and two 'soft' phases due to the symmetry $Z(K_1, K_2) = Z(K_2, K_1)$ of the partition function. The bifurcation at E is also revealed by the mean-field calculation of appendix 1 (see figure 6).

3.3. N = 6

The phase diagram is shown in figure 3, with

$$\begin{aligned} x_1 &= \exp[-(K_1/2+3K_2/2+2K_3)] \\ x_2 &= \exp[-\tfrac{3}{2}(K_1+K_2)] \\ x_3 &= \exp[-2(K_1+K_3)] \end{aligned} \tag{3.11}$$

with the self-dual line given by

$$\frac{x_1 - x_1^{\mathrm{V}}}{x_1^{\mathrm{P}} - x_1^{\mathrm{V}}} = \frac{x_2 - x_2^{\mathrm{V}}}{x_2^{\mathrm{P}} - x_2^{\mathrm{V}}} = \frac{x_3 - x_3^{\mathrm{V}}}{x_3^{\mathrm{P}} - x_3^{\mathrm{V}}} \tag{3.12}$$

where x_i^{P} and x_i^{V} are the critical points of the Potts and Villain models respectively.

Our insight into this model is aided by writing our $Z(6)$ variable S in terms of a $Z(2)$ variable σ and a $Z(3)$ variable Σ (Domany and Riedel 1979):

$$S = \sigma\Sigma. \tag{3.13}$$

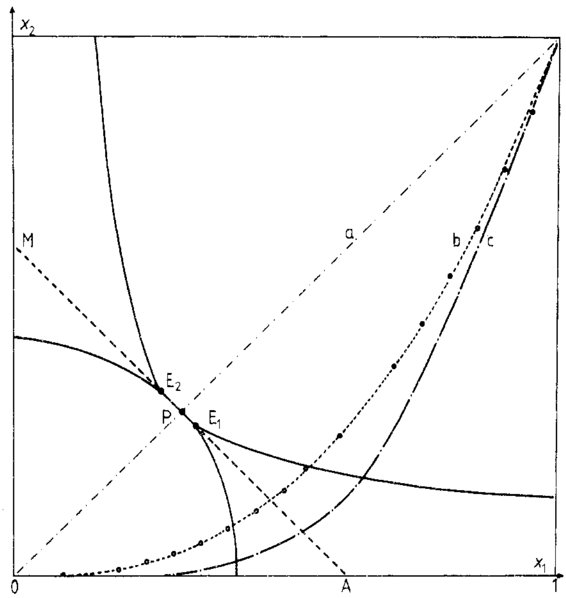

Figure 2. Schematic diagram of the $Z(5)$ model. The straight line AM is self-dual. The curves a, b and c represent the thermodynamic paths of the scalar Potts ($J_1 = J_2$), vector Potts ($J_2 = 0$) and Villain model respectively. P is the critical point of the five-state Potts model and E_1, E_2 are bifurcation points of the self-dual line at which the soft phases originate.

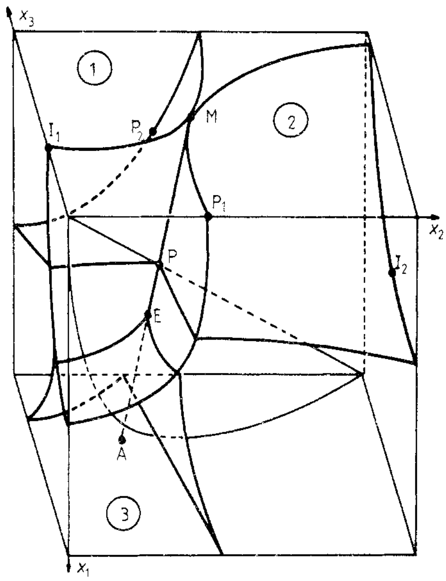

Figure 3. Schematic diagram of the $Z(6)$ model. The straight line AM is self-dual. The straight and curved lines going from (0, 0, 0) to (1, 1, 1) are the thermodynamic paths of the six-state Potts and Villain model respectively. The critical points are: P, six-state Potts; I_1, I_2, Ising; P_1, P_2, three-state Potts. The soft phase originates at E.

The action then becomes

$$A=-\sum_{\langle ij\rangle}\left[\frac{J_1}{2}[\sigma(i)\sigma(j)(\Sigma(i)\Sigma^+(j)+\mathrm{cc})-2]+\frac{J_2}{2}[\Sigma(i)\Sigma^+(j)+\mathrm{cc}-2]+J_3(\sigma(i)\sigma(j)-1)\right]. \tag{3.14}$$

The limits $J_2\to\infty$ ($x_1=x_2=0$) and $J_1=J_2=0$ ($x_2=1$, $x_1=x_3$) are Ising models with critical points $\mathrm{I}_1=(0,0,\sqrt{2}-1)$ and $\mathrm{I}_2=(\sqrt{2}-1,1,\sqrt{2}-1)$. The other limits $J_3\to\infty$ ($x_1=x_3=0$) and $J_1=J_3=0$ ($x_1=x_2$, $x_3=1$) are $Z(3)$ Potts models with critical points $\mathrm{P}_1=(0,(\sqrt{3}-1)/2,0)$ and $\mathrm{P}_2=((\sqrt{3}-1)/2,(\sqrt{3}-1)/2,1)$.

It is instructive to study the phases on the decoupling surface $J_1=0$ ($x_1=x_2x_3$), where we have two independent $Z(2)$ and $Z(3)$ models. In figure 4 we show the projection of this surface on the x_2, x_3 plane, where the straight lines $\overline{\mathrm{I}_1\mathrm{I}_2}$ and $\overline{\mathrm{P}_1\mathrm{P}_2}$ are the critical lines of the $Z(2)$ and $Z(3)$ models. All four phases are massive, because on this decoupling surface the correlation functions of the $Z(6)$ variables are products of $Z(2)$ and $Z(3)$ correlation functions, e.g.

$$\langle S(i)S(0)\rangle=\langle\sigma(i)\sigma(0)\rangle\langle\Sigma(i)\Sigma(0)\rangle \tag{3.15}$$

and everywhere except at the critical lines the right-hand side decays exponentially.

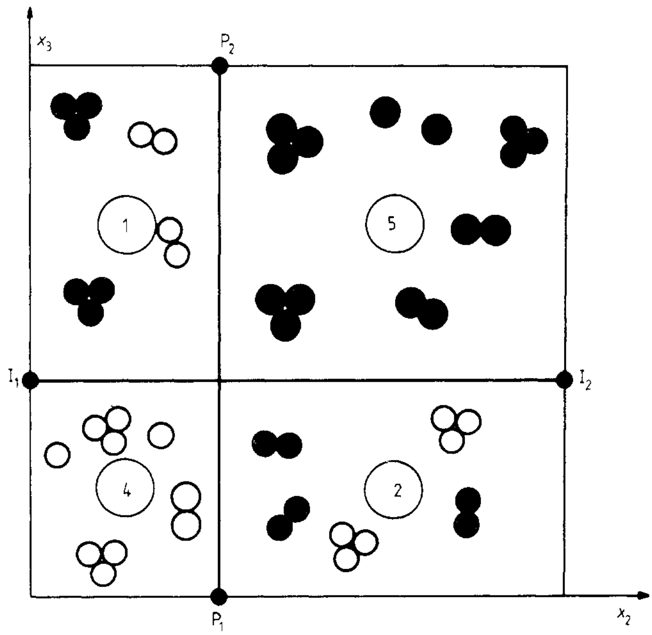

Figure 4. Projection of the decoupling surface onto the x_2, x_3 plane.

In the low-temperature phase 4 the $Z(6)$ symmetry is completely broken: $\langle S^n\rangle\neq 0$, $\langle\rho^n\rangle=0$; $n=1,2,3$. In the high-temperature phase 5 the symmetry is completely restored: $\langle S^n\rangle=0$, $\langle\rho^n\rangle\neq 0$; $n=1,2,3$. Phase 1 is $Z(2)$ invariant, because $\langle\sigma\rangle=0$, $\langle\Sigma\rangle\neq 0$ implying that $\langle S\rangle=0$, $\langle S^2\rangle\neq 0$, $\langle S^3\rangle=0$. Phase 2 is $Z(3)$ invariant because now $\langle\sigma\rangle\neq 0$, $\langle\Sigma\rangle=0$ implying that $\langle S\rangle=0$, $\langle S^2\rangle=0$, $\langle S^3\rangle\neq 0$. But our dual transformation $\mathscr{D}$ interchanges region 1 with region 2. Remembering that $\mathscr{D}$ also interchanges order and disorder, we find that in phase 1 $\langle\rho\rangle = 0$, $\langle\rho^2\rangle = 0$, $\langle\rho^3\rangle \neq 0$ and in phase 2 $\langle\rho\rangle = 0$, $\langle\rho^2\rangle \neq 0$, $\langle\rho^3\rangle = 0$. Notice that phases 1 and 2 are massive in spite of $\langle\rho\rangle = \langle S\rangle = 0$ and $[\rho, S] \neq 0$, because the vanishing of the order and disorder parameters is just a consequence of the $Z(2)$ and $Z(3)$ selection rules operating in phases 1 and 2. Thus no zero-mass states have to be invoked to explain the simultaneous vanishing of order and disorder. We summarise these results in table 1.

Table 1.

Phase	$\langle S\rangle$	$\langle S^2\rangle$	$\langle S^3\rangle$	$\langle\rho\rangle$	$\langle\rho^2\rangle$	$\langle\rho^3\rangle$
4	$\neq 0$	$\neq 0$	$\neq 0$	$= 0$	$= 0$	$= 0$
5	$= 0$	$= 0$	$= 0$	$\neq 0$	$\neq 0$	$\neq 0$
1	$= 0$	$\neq 0$	$= 0$	$= 0$	$= 0$	$\neq 0$
2	$= 0$	$= 0$	$\neq 0$	$= 0$	$\neq 0$	$= 0$

The complete phase diagram of figure 3 includes a phase in which $\langle S^n\rangle = 0$, $\langle\rho^n\rangle = 0$, $n = 1, 2, 3$, which now implies that this phase is soft. It is bounded by second- or higher-order lines due to its massless character. Since we know the Potts transition to be of first order, the 'soft' phase has to appear at some position E ($\neq$P), which can be estimated by mean-field calculations (appendix 1).

3.4. $N = 7$

In figure 5 we show the phase diagram for $N = 7$, exhibiting spontaneous $Z(7)$ breaking along the three straight lines and three 'soft' phases. This threefoldness is a consequence of the cyclic symmetry of the partition function in K_1, K_2, K_3.

4. $Z(N)$ gauge system in four dimensions

This section is dedicated to an analysis of the phase diagrams of four-dimensional $Z(N)$ gauge systems. A large number of structural analogies between spin systems in two dimensions and gauge systems in four dimensions have been revealed (Kogut 1979), so that the argument of this section would be expected to be similar to that of § 3.

To construct a system with a local $Z(N)$ gauge symmetry (Wegner 1972) on a four-dimensional hypercubic lattice we place a $Z(N)$ variable $S(r, \mu) = S(r+\mu, -\mu)$ at each link. Here r indicates a lattice point and μ the direction of the link. Our action is to be invariant under a local gauge transformation at site r defined by the operation $G(r)$ of rotating all links emanating from that site through $2\pi/N$ ($-2\pi/N$), if the links are parallel (antiparallel) to the basis vectors.

The simplest term with this symmetry is the 'plaquette' defined as the oriented product of links around an elementary square

$$A_P = S(r, \mu)S(r+\mu, \nu)S^+(r+\mu+\nu, \mu)S^+(r+\nu, -\nu). \tag{4.1}$$

The most general locally $Z(N)$-invariant action of 'ferromagnetic' type is then

$$A = \sum_P \sum_{\alpha=1}^{\bar{N}} \frac{J_\alpha}{2}([A_P]^\alpha + \mathrm{cc} - 2) \tag{4.2}$$

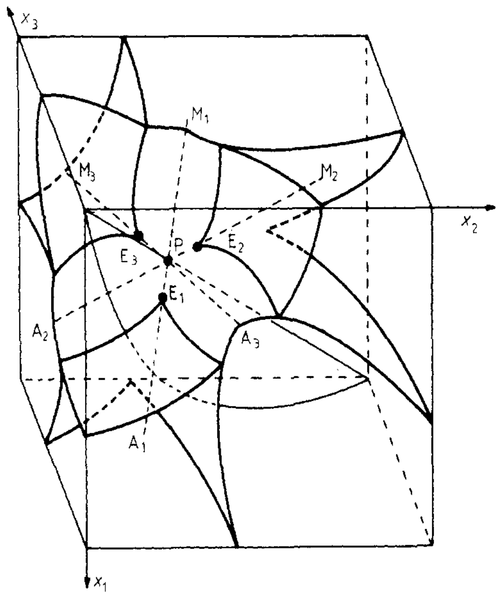

Figure 5. Schematic diagram of $Z(7)$ model. The straight lines A_1M_1, A_2M_2, A_3M_3 are self-dual. The straight and curved lines going from (0, 0, 0) to (1, 1, 1) are the thermodynamic paths of the seven-state Potts and Villain model respectively. P is the seven-state Potts critical point and E_1, E_2, E_3 are soft-phase bifurcation points.

where Σ_P extends over all plaquettes of the lattice. If only J_1 is non-vanishing we obtain Wilson's action (Wilson 1974), which has been studied by many authors in the Euclidean (Ukawa *et al* 1980) and Hamiltonian formulation (Horn *et al* 1979).

As in the spin case we shall introduce disorder variables and the same duality transformation. For this purpose we reduce the variables to a minimum number of independent ones. First we choose the temporal gauge

$$S(r, \hat{4}) = 1 \tag{4.3}$$

in which the action becomes

$$A = -\sum_{x_4=-M/2}^{+M/2} \sum_{x_1,x_2,x_3} \left\{ \sum_i \sum_{\alpha=1}^{\bar{N}} \frac{J_\alpha}{2} [(S(x_1x_2x_3x_4; \hat{\imath})S^+(x_1x_2x_3x_4+1; \hat{\imath}))^\alpha + \text{CC} - 2] \right.$$
$$\left. + \sum_{\substack{i,j \\ i<j}} \sum_{\alpha=1}^{\bar{N}} \frac{J_\alpha}{2} [(S(r; \hat{\imath})S(r+\hat{\imath}; \hat{\jmath})S^+(r+\hat{\jmath}; \hat{\imath})S^+(r; \hat{\jmath}))^\alpha + \text{CC} - 2] \right\}. \tag{4.4}$$

As in the spin case the corresponding transfer matrix is

$$T = \prod_r \prod_{\substack{i,j \\ i<j}} \exp\left\{ \sum_{\alpha=1}^{\bar{N}} \frac{K_\alpha}{2} [(S(r; \hat{\imath})S(r+\hat{\imath}, \hat{\jmath})S^+(r+\hat{\jmath}; \hat{\imath})S^+(r; \hat{\jmath}))^\alpha + \text{HC} - 2] \right\}$$
$$\times \prod_r \prod_i \exp\left\{ \sum_{\alpha=0}^{\bar{N}} \frac{f_\alpha}{2} [(R(r; \hat{\imath}))^\alpha + \text{HC}] \right\} \tag{4.5}$$

where $S(\boldsymbol{r}, i)$ are now unitary operators and $R(\boldsymbol{r}, i)$ are link rotation operators satisfying the algebra (2.9) and f_α are defined by equation (2.12).

Since the temporal gauge does not fix the configurations on the gauge-invariant subspace

$$G(r)|\rangle = |\rangle \tag{4.6}$$

we use the identity

$$1 = G(r) = R(r; \hat{1})R^+(r; -\hat{1})R(r; \hat{2})R^+(r; -\hat{2})R(r; \hat{3})R^+(r; -\hat{3}) \tag{4.7}$$

to eliminate further dependent variables, namely

$$R^+(x_1x_2x_3; \hat{3}) = \prod_{n\geq 0} \prod_{i=1}^{2} R(x_1x_2x_3 - n; i)R^+(x_1x_2x_3 - n; -i). \tag{4.8}$$

Since now all terms commute with $S(x_1, x_2, x_3; \hat{3})$ we may choose the axial gauge

$$S(r; \hat{3}) = 1.$$

For these independent variables we now introduce dual variables existing on the dual lattice with sites

$$\tilde{r} = r + (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}). \tag{4.9}$$

Notice that in the transfer matrix formulation, we introduced operators existing on a three-dimensional lattice.

The dual partner of the plaquette variable (2-simplex) is a link variable (1-simplex) of the dual lattice

$$\sigma(\tilde{r}, i) = S(r + \hat{\imath}; \hat{\jmath})S(r + \hat{\imath} + \hat{\jmath}; \hat{k})S^+(r + \hat{\imath} + \hat{k}; \hat{\jmath})S^+(r + \hat{\imath}; \hat{k}) \tag{4.10}$$

where i, j, k are cyclic permutations of 1, 2, 3. The non-local dual disorder variables are

$$\begin{aligned} \rho(\tilde{r}; \hat{1}) &= \prod_{n\geq 0} R^+(x_1 + 1, x_2, x_3 - n; \hat{2}) \\ \rho(\tilde{r}; \hat{2}) &= \prod_{n\geq 0} R(x_1, x_2 + 1, x_3 - n; \hat{1}). \end{aligned} \tag{4.11}$$

These ρ variables associated with an elementary square perpendicular to the $\hat{3}$ direction, which defines a link on the direct lattice, are used to obtain $R^+(r, \hat{3})$:

$$R^+(r; \hat{3}) = \rho(\tilde{r} - \hat{1} - \hat{2}; \hat{1})\rho(\tilde{r} - \hat{2}; \hat{2})\rho^+(\tilde{r}; -\hat{1})\rho^+(\tilde{r} - \hat{1}; -\hat{2}). \tag{4.12}$$

The analogous expressions for $R^+(r, \hat{1})$ and $R^+(r, \hat{2})$ are, in the axial gauge for the dual variables $\rho(\tilde{r}; \hat{3}) = 1$,

$$R^+(r; \hat{1}) = \rho(\tilde{r} - \hat{3}; -\hat{2})\rho^+(\tilde{r} - \hat{2}; \hat{2}) \tag{4.13}$$

$$R^+(r; \hat{2}) = \rho^+(\tilde{r} - \hat{3}; -\hat{1})\rho(\tilde{r} - \hat{1}; \hat{1}). \tag{4.14}$$

We easily verify that ρ and σ satisfy the same algebra as S and R. Expressing the transfer matrix in terms of them, we obtain

$$\begin{aligned} T = \prod_{\tilde{r}} \prod_{i} \exp\left\{ \sum_{\alpha=1}^{\tilde{N}} \frac{K_\alpha}{2}[(\sigma(\tilde{r}; i))^\alpha + \mathrm{HC} - 2]\right\} \\ \times \prod_{\tilde{r}} \prod_{\substack{i,j \\ i<j}} \exp\left\{ \sum_{\alpha=0}^{\tilde{N}} \frac{f_\alpha}{2}[(\rho(\tilde{r}; \hat{\imath})\rho(\tilde{r} + \hat{\imath}; \hat{\jmath})\rho^+(\tilde{r} + \hat{\jmath}; \hat{\imath})\rho^+(\tilde{r}; \hat{\jmath}))^\alpha + \mathrm{HC} - 2]\right\} \end{aligned} \tag{4.15}$$

yielding the same duality relation (equation (2.18)) as in the spin case

$$Z(\{K_\alpha\},\{f_\beta\}) = eZ(\{f_\alpha\},\{K_\beta\}). \tag{4.16}$$

The reasoning following equation (2.20) also applies here. As in the spin case the Potts ($x_1 = x_2 = \ldots = x_N$) and Villain models are self-dual in the Kramers–Wannier sense. The Wilson model ($J_2 = J_3 = \ldots = J_{\tilde{N}} = 0$) obeys this duality only for $N < 5$. (In the time-continuous Hamiltonian formulation (Horn *et al* 1979) this model is self-dual for any N, because in the limiting procedure involved to make the time variable continuous the terms violating self-duality go to zero.)

Let us now analyse the phase diagram of our gauge systems. We conjecture that these diagrams appear exactly as the corresponding ones for the two-dimensional spin systems. Only the characterisation of the phases and the order of the transition changes. This expectation is borne out by mean-field calculations given in appendix 1. Thus one would naively expect that at low temperatures, where the two-dimensional system is in the ordered phase and $\langle S\rangle \neq 0$, the correlation functions of the gauge system fall off slowly, that is, the Wilson loop

$$A(C) = \prod_C S \tag{4.17}$$

(where C is a closed curve of links on the lattice) decays as the perimeter of C. In this phase the disorder parameter of the spin system has vanishing average $\langle\rho\rangle = 0$, so that the corresponding gauge system variable, 't Hooft's disorder loop

$$B(\tilde{C}) = \prod_{\tilde{C}} \rho \tag{4.18}$$

(where $\tilde{C}$ is a closed curve of links on the dual lattice) should decay much faster, namely as the minimal area enclosed by $\tilde{C}$. Thus one obtains the correspondence

$$\begin{aligned} &\langle S\rangle \neq 0 \leftrightarrow A(C) \approx \exp(-P) \qquad &&\langle S\rangle = 0 \leftrightarrow A(C) \approx \exp(-A) \\ &\langle\rho\rangle \neq 0 \leftrightarrow B(\tilde{C}) \approx \exp(-P) \qquad &&\langle\rho\rangle = 0 \leftrightarrow B(\tilde{C}) \approx \exp(-A) \end{aligned} \tag{4.19}$$

which, as we shall see, is not completely correct. For example, we showed in § 3 that we cannot have simultaneously $\langle S^n\rangle \neq 0$ and $\langle\rho^k\rangle \neq 0$ in the same spin phase, unless S^n and ρ^k commute. On the other hand, for gauge systems we do obtain phases in which both $A(C)$ and $B(\tilde{C})$ obey area and perimeter decays (this occurs for $N = 6$).

Looking now for particular values of N we obtain the following picture.

4.1.

For $N = 2, 3$ our action coincides with Wilson's, since there is only one coupling constant. Monte Carlo calculations (Creutz *et al* 1979) and heuristic arguments indicate that this is a two-phase system (magnetic and electric confinement phase) separated by a first-order transition. Our duality equation (2.20) locates these at (Yoneya 1978):

$$\begin{aligned} &N = 2\colon 1 + x_1 = \sqrt{2}, \qquad &&K_1^C = \tfrac{1}{2}\ln(1+\sqrt{2}) \\ &N = 3\colon 1 + 2x_1 = \sqrt{3}, \qquad &&K_1^C = \tfrac{2}{3}\ln(1+\sqrt{3}) \end{aligned} \tag{4.20}$$

with area decay at high temperatures and perimeter fall off at low temperatures for the Wilson loop (and opposite behaviour for 't Hooft's loop).

4.2.

For $N=4$ our parameter space is two-dimensional, with the self-dual line given by

$$1+2x_1+x_2=\sqrt{4}; \qquad \begin{aligned} x_1 &= \exp[-(K_1+2K_2)] \\ x_2 &= \exp[-2K_1]. \end{aligned} \tag{4.21}$$

If a given model (x_1, x_2) suffers only one phase transition, its critical point is located at the intersection of equation (4.21) with the model's thermodynamic path Γ.

To gain some insight into the phase diagram let us proceed as in the spin model and look at limiting cases. To this end we write the $Z(4)$ variable in terms of two $Z(2)$ variables as in equation (3.6). The action then becomes

$$\begin{aligned} A_4 = -\frac{J_1}{4}\sum_P [(&\sigma(1)\sigma(2)\sigma(3)\sigma(4)+\sigma(1)\tau(2)\sigma(3)\tau(4) \\ &-\sigma(1)\tau(2)\tau(3)\sigma(4)-\sigma(1)\sigma(2)\tau(3)\tau(4))+(\sigma\leftrightarrow\tau)] \\ &-J_2\sum_P \sigma(1)\sigma(2)\sigma(3)\sigma(4)\tau(1)\tau(2)\tau(3)\tau(4)+\text{constant}. \end{aligned} \tag{4.22}$$

It is now evident that $J_1\to 0$ $(x_2\to 1)$ gives a $Z(2)$ gauge model for the variable $S^2=\sigma\tau$ with Ising critical temperature $x_1^c=\sqrt{2}-1$. Here $\langle A^2(C)\rangle$ changes from perimeter to area decay. The dually transformed model $\tilde{x}_1=0$ $(J_2\to\infty)$ must also be a $Z(2)$ gauge theory with critical point $x_2^c=\sqrt{2}-1$. Appealing to our continuity hypothesis we now continue these two critical points into the interior of the phase diagram. (The stability of these points, when we depart from the lines $x_1=0$ and $x_2=1$, together with the characterisation of the resulting phases given below, can be checked by high- and low-temperature perturbation theory. This can be done for any N.) $1/N$ expansions now indicate (Pearson *et al* 1980) that all Potts gauge models have a first-order transition and we will assume that this transition is unique (located at $x_1^c=x_2^c=\ldots=x_N^c=(\sqrt{N}+1)^{-1}$ from duality). Furthermore, Monte Carlo experiments also show only one first-order transition for Wilson's $Z(4)$ model. Thus we conclude that a bifurcation occurs, which we locate at the Potts point by analogy with the spin case. The conjectured phase diagram is shown in figure 1, where the phases are characterised as follows.

Phase 1 (magnetic confining phase):

$$\begin{aligned} \langle A^n(C)\rangle &\sim \exp(-P) \\ \langle B^n(\tilde{C})\rangle &\sim \exp(-A) \end{aligned} \qquad n=1,2.$$

Phase 2 (electric confining phase):

$$\begin{aligned} \langle A^n(C)\rangle &\sim \exp(-A) \\ \langle B^n(\tilde{C})\rangle &\sim \exp(-P) \end{aligned} \qquad n=1,2.$$

Phase 3:

$$\begin{aligned} \langle A(C)\rangle &\sim \exp(-A) \\ \langle A^2(C)\rangle &\sim \exp(-P) \end{aligned}$$

(only fundamental charges confined)

$$\langle B(\tilde{C})\rangle \sim \exp(-A)$$

$$\langle B^2(\tilde{C})\rangle \sim \exp(-P)$$

(no massless particles present in this phase).

Since $[A^2, B^2] = 0$ 't Hooft's argument in favour of massless particles does not apply in phase 3. In fact it would be very difficult to reconcile a soft phase limited by first-order phase boundaries. (Recall that at first-order phase transitions no long-range effects exist, which would be required by massless excitations.) In phase 3 the fundamental electric and magnetic charges are confined, whereas states created by S^2 and ρ^2 appear asymptotically.

4.3.

For $N = 5$ we have a two-dimensional coupling constant space with the self-dual line $1 + 2x_1 + 2x_2 = \sqrt{5}$.

As in the spin case the $Z(5)$ algebra of $A(C)$ and $B(\tilde{C})$ requires a soft phase (Elitzur *et al* 1979) in the region $x_1 > x_2$, and we again suppose bifurcation to occur at some point E different from the Potts transition. Due to its massless character this phase must be bounded by second- (or higher-) order phase transition lines, whereas at $\overline{\mathrm{EP}}$ the transition is first order (Creutz *et al* 1979). The three phases are characterised as

Phase 1:	$A^n(C) \approx \exp(-P)$,	$B^k(\tilde{C}) \approx \exp(-A)$	$m \neq 0$
Phase 2:	$A^n(C) \approx \exp(-A)$,	$B^k(\tilde{C}) \approx \exp(-P)$	$m \neq 0$
Phase 3:	$A^n(C) \approx \exp(-P)$,	$B^k(\tilde{C}) \approx \exp(-P)$	$m = 0$
	$n, k = 1, 2,$		

and are shown in figure 2.

4.4.

$N = 6$ is a rather interesting and instructive example of possible situations that may occur. The parameter space is three-dimensional with x_i given by equation (3.11), the self-dual line going through the Potts and Villain critical points (see figure 3).

As in the spin case (equation (3.14)) we write the $Z(6)$ variable as a product of $Z(2)$ and $Z(3)$ variables, with the action taking the form

$$A_6 = -\sum_P \left[\frac{J_1}{2}\sigma(1)\sigma(2)\sigma(3)\sigma(4)(\Sigma(1)\Sigma(2)\Sigma^+(3)\Sigma^+(4) + \mathrm{cc}) + \frac{J_2}{2}(\Sigma(1)\Sigma(2)\Sigma^+(3)\Sigma^+(4) + \mathrm{cc}) + J_3\sigma(1)\sigma(2)\sigma(3)\sigma(4)\right] + \text{constant}. \tag{4.23}$$

The same decoupling as in the spin case holds here on the surface $x_1 = x_2x_3$ ($J_1 = 0$). We obtain the same projection on the x_2, x_3 plane shown in figure 4. The critical lines correspond now to first-order transitions (Creutz *et al* 1979). Since $S(i)$ correlation functions are products of $\sigma(i)$ and $\Sigma(i)$ correlation functions (as in the spin case equation (3.15)) we obtain the result shown in table 2, where we used that $\langle \Pi_C \sigma\rangle \sim \exp(-P)$ for $T < T_c$, $\langle \Pi_C \sigma\rangle \sim \exp(-A)$ for $T > T_c$, and analogously for Σ.

Table 2.

Phase	$\langle A(C)\rangle$	$\langle A^2(C)\rangle$	$\langle A^3(C)\rangle$	$\langle B(\tilde{C})\rangle$	$\langle B^2(\tilde{C})\rangle$	$\langle B^3(\tilde{C})\rangle$	Condensate
4	$\exp(-P)$	$\exp(-P)$	$\exp(-P)$	$\exp(-A)$	$\exp(-A)$	$\exp(-A)$	e
5	$\exp(-A)$	$\exp(-A)$	$\exp(-A)$	$\exp(-P)$	$\exp(-P)$	$\exp(-P)$	m
1	$\exp(-A)$	$\exp(-P)$	$\exp(-A)$	$\exp(-A)$	$\exp(-A)$	$\exp(-P)$	$2e, 3m$
2	$\exp(-A)$	$\exp(-A)$	$\exp(-P)$	$\exp(-A)$	$\exp(-P)$	$\exp(-A)$	$3e, 2m$

For the dynamical mechanism producing this result we propose the following. In phase 4 (phase 5) we have a condensate of single, double and triple charges (monopoles) confining the monopoles (charges). Phases 1 and 2 have a more complicated structure.

In 'superconducting language' we may state the following. In phase 4 (phase 5) we have a condensate of all charges (monopoles) confining the monopoles (charges). In phase 1 the condensate consists of double charges and triple monopoles, which is allowed by the perimeter decay of $\langle A^2(C)\rangle$ and $\langle B^3(\tilde{C})\rangle$. This condensate confines the other charges and monopoles. Whether the dynamical mechanism is really confinement or bleaching (Rothe *et al* 1979) can only be verified by examining correlation functions for small distances or coupling additional quantum numbers, whose emergence can be checked.

4.5.

For $N = 7$ we show the phase diagram in figure 5. *Mutatis mutandis* we have the same interpretation as for $N = 5$.

5. Conclusion

We have proposed very plausible phase diagrams for two-dimensional spin and four-dimensional gauge systems. These appear identical in the two cases, differing only in the order of the transitions. We exhibit a mean-field calculation where this is true. The order of the gauge system transition is always lower than that of the corresponding spin transition. The phases of the spin system are characterised by the behaviour of the various powers of order and disorder parameters. The simultaneous non-vanishing of both is forbidden unless they commute. For gauge systems the phases are characterised by the decay properties of powers of the Wilson and 't Hooft loops. All combinations of perimeter and area decay of these loops are allowed.

Acknowledgments

After the completion of this paper we became aware of a paper by Cardy (1980) which treats two-dimensional $Z(N)$ spin systems with results very similar to ours.

It is a great pleasure to thank J A Swieca for many helpful and stimulating discussions throughout this work. Professor Jorge Andre Swieca died unexpectedly on December 22, 1980. As a small tribute to his readiness in sharing his insights accomplished by a superb intellect, we dedicate this paper to his memory.

Appendix 1. Mean-field approximation

The mean-field approximation (MFA) neglects fluctuations, reducing computations to those of a one-body problem. For that purpose an external field H is coupled to the local order parameter field and H is determined self-consistently. Since such an order parameter does not exist for systems with a local gauge symmetry, applying MFA in this case is problematic. In the following we briefly describe the well known MFA for spin systems in order to compare it with results for gauge systems.

A1.1. Spin models

The partition function for a lattice with M sites is

$$Z=\exp(-M\beta f)=\prod_{i=1}^{M}\sum_{S(i)}\exp\left\{\beta\sum_{\langle i,j\rangle}\left[\sum_{\alpha=1}^{\bar{N}}\frac{J_\alpha}{2}((S(i)S^+(j))^\alpha+\mathrm{cc})\right]\right\}. \qquad (A1.1)$$

Following Balian *et al* (1975) we define the uncorrelated measure

$$D[S_i]=\frac{\prod_{i=1}^{M}\exp\left[\frac{H}{2}(S_{(i)}+S^*_{(i)})\right]}{\prod_{i=1}^{M}\sum_{S_i=0}^{N-1}\exp\left[\frac{H}{2}(S_{(i)}+S^*_{(i)})\right]} \qquad (A1.2)$$

to be used in the convexity inequality of the exponential function

$$\langle\exp(A)\rangle\geqslant\exp(\langle A\rangle). \qquad (A1.3)$$

Taking for A the expression

$$\exp\left\{\beta\sum_{\langle i,j\rangle}\sum_{\alpha=1}^{\bar{N}}\frac{J_\alpha}{2}[(S_{(i)}S^*_{(j)})^\alpha+\mathrm{cc}]-\sum_i\frac{H}{2}(S_{(i)}+S^*_{(i)})\right\} \qquad (A1.4)$$

and remembering that we have M sites and Md links of nearest-neighbour pairs we obtain

$$-\beta f\geqslant\mathscr{F}_N(H)-H\mathscr{F}'_N(H)+\beta d\sum_{\alpha=1}^{\bar{N}}J_\alpha t_\alpha^2(H)\equiv G_N(H,\beta) \qquad (A1.5)$$

where

$$\mathscr{F}_N(H)=\ln\left\{\sum_{n=0}^{N-1}\exp\left[H\cos\frac{2\pi}{N}n\right]\right\},\qquad \mathscr{F}'_N(H)=\frac{\mathrm{d}}{\mathrm{d}H}\mathscr{F}_N(H)$$

$$t_\alpha(H)=\left[\sum_{n=0}^{N-1}\cos\frac{2\pi}{N}n\alpha\exp\left(H\cos\frac{2\pi}{N}n\right)\right]\Big/\left[\sum_{n=0}^{N-1}\exp\left(H\cos\frac{2\pi}{N}n\right)\right].$$

For a given β we determine H by maximising the right-hand side of (A1.5):

$$-\beta f\geqslant\sup_H(G_N(H,\beta)). \qquad (A1.6)$$

With this scheme we obtained the following results:

(i) All Potts models for $N\geqslant 3$ have one first-order transition.

(ii) For $N\geqslant 5$ we applied MFA to models whose Γ crosses the self-dual surface between P and E (see figures 2, 3, 5 and 6). As we move from P to E the latent heat diminishes and at the same time a second maximum of $G_N(H)$ appears, indicating a bifurcation of Σ at E. In figure 6, curve c, we show such a result for a $Z(5)$ model with $J_1 = 1$, $J_2 = 0.45$. There is a first-order transition around $\beta_c d = 1.028$ and a second-order one at $\beta_c d = 1.000$. This last value seems to be independent of N. Notice that for a MFA to show two transitions, one of them must be first order. In reality this is a defect of the approximation and both transitions are second- (or higher-) order, because the intermediate phase is soft.

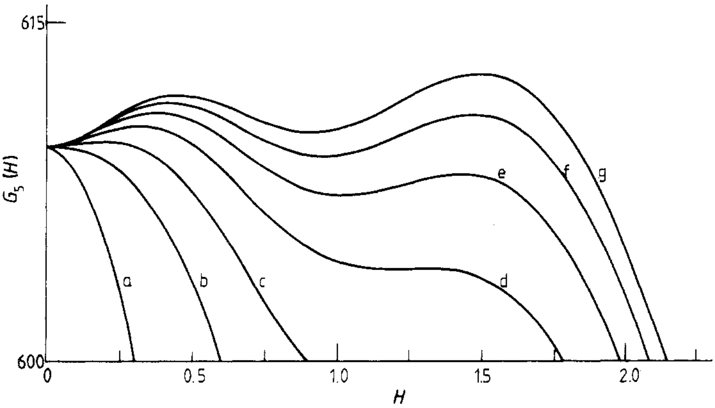

Figure 6. The maximum of the function $G_5(H)$ gives for each β the free energy per particle $(-\beta f)$ in MFA. The curves correspond to $J_1 = 1$ and $J_2 = 0.45$. The curves a to g correspond to βd equal to 0.90, 0.99, 1.01, 1.02, 1.026, 1.028 and 1.030 respectively. For each β the maximum gives the free energy of the model in the mean-field approximation. One transition occurs at $\beta_1 d \sim 1.000$ and the second one at $\beta d_2 \sim 1.028$.

A1.2. Gauge models

The issue here is the choice of measure to be used in inequality (A1.3). Since the physics is gauge invariant it would seem natural to use a gauge-invariant measure, for example

$$D_1[S(r,\mu)] = \frac{\prod_{r,i} \exp\left[\frac{H}{2}(S(1)S(2)S^+(3)S^+(4) + \mathrm{cc})\right]}{\prod_{r,i} \sum_{S(SV(r,i))} \exp\left[\frac{H}{2}(\sum_{P^*} (S(1)S(2)S^+(3)S^+(4) + \mathrm{cc})\right]} \tag{A1.7}$$

where each link belongs to only one plaquette out of a subset P^* of plaquettes as shown in figure 7. But since we know that a local symmetry cannot be broken spontaneously (Elitzur 1975, Lüscher 1977), we expect this MFA to show no phase transition at all. In fact our calculations with $D_1[S(r,\mu)]$ show only one phase.

We could also follow Balian *et al* (1978) and Drouffe and Itzykson (1978) and proceed as in the spin case using

$$D_2[S(r,\mu)] = \frac{\prod_{r,\mu} \exp\left[\frac{H}{2}(S(r,\mu) + S^+(r,\mu))\right]}{\prod_{r,\mu} \sum_{S(r,\mu)} \exp\left[\frac{H}{2}(S(r,\mu) + S^+(r,\mu))\right]}. \tag{A1.8}$$

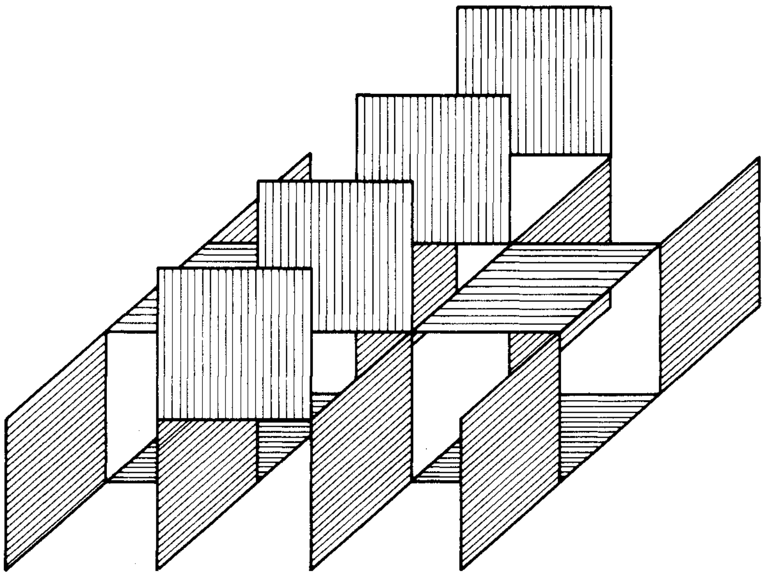

Figure 7. Set of plaquettes, such that each link belongs only to one plaquette.

But with this measure, violating gauge invariance, we obtain only one first-order phase transition for all N. In figure 8 we show a typical result for Wilson's $Z(6)$ model ($J_2 = J_3 = 0$).

Faced with this dilemma we make a compromise and use a measure which breaks gauge invariance on only half the total number of sites:

$$D_3[S(r, \mu)] = \frac{\prod_{P^*} \exp\left[\frac{H}{2}(S(1)S^+(4) + S(2)S^+(3) + \text{cc})\right]}{\sum_{S(r,\mu)} \exp\left[\frac{H}{2}\sum_{P^*}(S(1)S^+(4) + S(2)S^+(3) + \text{cc})\right]}. \tag{A1.9}$$

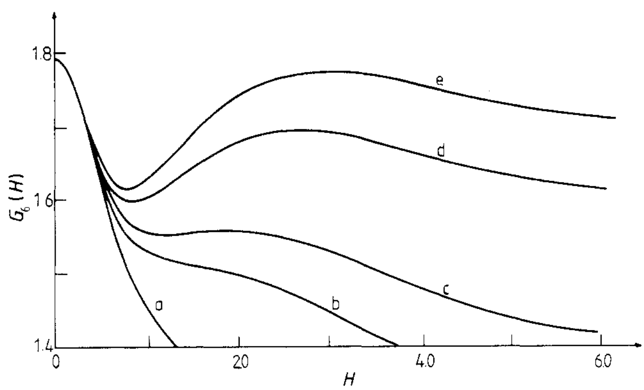

Figure 8. $G_6(H)$ versus effective field for Wilson's $Z(6)$ model with the measure (A1.8). The curves a to e correspond to $J_1\beta(d-1)$ equal to 1.0, 1.3, 1.4, 1.6 and 1.7 respectively ($J_2 = J_3 = 0$).

Proceeding as in the spin case and remembering that the number of plaquettes belonging to P^* is $Md/4$ we obtain

$$-\beta f \geqslant \frac{d}{2}\left(\ln N + \mathscr{F}_N(H) - H\mathscr{F}'(H) + \frac{\beta}{2}\sum_{\alpha=1}^{\bar{N}} J_\alpha t_\alpha^2(H)\right) \qquad (A1.10)$$

which is identical to the two-dimensional spin result (A1.5) up to a factor $d/2 = 2$ and a different inverse temperature β. With this form of MFA we will thus obtain a diagram with the same number of phases as for the spin system. We feel that this last type of MFA is nearer to the truth than the other two.

Appendix 2. Gauge theory with Higgs fields in three dimensions

Since in three dimensions the duality transformation takes a plaquette into a site we have to add Higgs fields $\phi(r)$ defined on sites in order to obtain a self-dual system. The most general $Z(N)$-invariant action is now given by

$$A = -\sum_P \sum_{\alpha=1}^{\bar{N}} \frac{J_\alpha}{2}[(A_P)^\alpha + \text{CC} - 2] - \sum_{r,i}\sum_{\alpha=1}^{\bar{N}}\left[\frac{K_\alpha}{2}(\phi^+(r)S(r;\hat{\imath})\phi(r+\hat{\imath}))^\alpha + \text{CC} - 2\right] \qquad (A2.1)$$

where A_P are plaquette variables and $\phi(r)$ are matter fields satisfying

$$[\phi(r)]^N = 1.$$

Proceeding as in § 4 we obtain the transfer matrix in the temporal gauge

$$T = \prod_{r,i}\exp\left\{\beta\sum_{i<j\leqslant 2}\sum_{\alpha=1}^{\bar{N}}\frac{J_\alpha}{2}[(S(r;\hat{\imath})S(r+\hat{\imath};\hat{\jmath})S^+(r+\hat{\jmath};\hat{\imath})S^+(r;\hat{\jmath}))^\alpha + \text{HC} - 2]\right.$$
$$\left. + \beta\sum_{\alpha=1}^{\bar{N}}\frac{K_\alpha}{2}[(\phi^+(r)S(r;\hat{\imath})\phi(r+\hat{\imath}))^\alpha + \text{HC} - 2]\right\}$$
$$\times\prod_r \exp\left\{\beta\sum_{i=1}^{2}\sum_{\alpha=0}^{\bar{N}}\frac{f_\alpha}{2}(R^\alpha(r;\hat{\imath}) + \text{HC}) + \beta\sum_{\alpha=0}^{\bar{N}}\frac{g_\alpha}{2}[B^\alpha(r) + \text{HC}]\right\} \qquad (A2.2)$$

where $R(r, \hat{\imath})$ and $B(r)$ are rotation operators for link and site variables satisfying the algebra (2.9). The functions f_α are defined by equation (2.12) and g_α analogously by

$$\sum_{\alpha=0}^{N-1} B^\alpha \exp\left[\beta\sum_{\eta=1}^{\bar{N}} K_\eta\left(\cos\frac{2\pi}{N}\alpha\eta - 1\right)\right] = \exp\left[\sum_{\alpha=0}^{\bar{N}}\frac{g_\alpha}{2}(B^\alpha + \text{CC})\right]. \qquad (A2.3)$$

Working in the gauge-invariant subspace and selecting the axial gauge

$$S(r, 2) = 1 \qquad (A2.4)$$

we introduce variables dual to plaquettes

$$\mu_1(\tilde{r}) = S(r, \hat{1})S(r+\hat{1}, \hat{2})S^+(r+\hat{2}, \hat{1})S^+(r, \hat{2}) \qquad (A2.5)$$

existing on sites $\tilde{r}$ and variables dual to sites

$$\begin{aligned}\mu_1(\tilde{r}, \hat{2}) &= \phi^+(r+\hat{2})S(r+\hat{2}, \hat{1})\phi(r+\hat{1}+\hat{2})\\ \mu_1(\tilde{r}, \hat{1}) &= \phi^+(r+\hat{1})S(r+\hat{1}, \hat{2})\phi(r+\hat{1}+\hat{2})\end{aligned} \qquad (A2.6)$$

existing on links of the dual lattice. The non-local disorder variables are now

$$\mu_3(\tilde{r}) = \prod_{n\geq 0} R(x, y-n, \hat{1})$$
$$\mu_3(\tilde{r}, \hat{1}) = \prod_{n\geq 0} B(x+\hat{1}, y-n). \tag{A2.7}$$

From equation (A2.4) it follows that

$$\mu_3(\tilde{r}, \hat{2}) = 1. \tag{A2.8}$$

The rotation operators can now be expressed in terms of the disorder variables as

$$R(r, \hat{2}) = \mu_3^+(\tilde{r})\mu_3^+(\tilde{r}-\hat{1}, \hat{1})\mu_3(\tilde{r}-\hat{1})$$
$$R^+(r, \hat{1}) = \mu_3^+(\tilde{r})\mu_3^+(\tilde{r}-\hat{2}, \hat{2})\mu_3(\tilde{r}-\hat{2}) \tag{A2.9}$$
$$B^+(r) = \mu_3(\tilde{r}-\hat{1}-\hat{2}, \hat{1})\mu_3(\tilde{r}-\hat{2}, \hat{2})\mu_3^+(\tilde{r}-\hat{1}, \hat{1})\mu_3^+(\tilde{r}-\hat{1}-\hat{2}, \hat{2}).$$

One easily verifies that the dual variables satisfy the same algebra as the original ones. In terms of the dual variables the transfer matrix becomes

$$T = \prod_r \exp \beta\Big[\sum_{\alpha=1}^{\bar{N}} \frac{J_\alpha}{2}(\mu_1^\alpha(\tilde{r}) + \text{HC} - 2) + \sum_{i=1}^{2}\sum_{\alpha=1}^{\bar{N}} \frac{K_\alpha}{2}(\mu_1^\alpha(\tilde{r}, \hat{\imath}) + \text{HC} - 2)\Big]$$
$$\times \prod_{r,i} \exp \beta\Big\{\sum_{\alpha=0}^{\bar{N}} \frac{f_\alpha}{2}[(\mu_3^+(\tilde{r})\mu_3(\tilde{r}, \hat{\imath})\mu_3(\tilde{r}+\hat{\imath}))^\alpha + \text{HC}]$$
$$+ \sum_{i<j\leq 2}\sum_{\alpha=0}^{\bar{N}} \frac{g_\alpha}{2}[(\mu_3(\tilde{r}, \hat{\imath})\mu_3(\tilde{r}+\hat{\imath}, \hat{\jmath})\mu_3^+(\tilde{r}+\hat{\jmath}, \hat{\imath})\mu_3^+(\tilde{r}, \hat{\jmath}))^\alpha + \text{HC}]\Big\}, \tag{A2.10}$$

yielding the following self-duality relation when compared with (A2.2):

$$Z(\{J_\alpha\}, \{K_\alpha\}, \{f_\alpha\}, \{g_\alpha\}) = eZ(\{g_\alpha\}, \{f_\alpha\}, \{K_\alpha\}, \{J_\alpha\}). \tag{A2.11}$$

Our duality transformation $\mathscr{D}: J_\alpha, K_\alpha \to g_\alpha, f_\alpha$, is simplified when written in terms of the parameters

$$x_\alpha \equiv \exp \sum_{\delta=1}^{\bar{N}} \beta J_\delta\Big[\cos\Big(\frac{2\pi\alpha\delta}{N}\Big) - 1\Big], \qquad x_{N-\alpha} = x_\alpha$$
$$y_\alpha \equiv \exp \sum_{\delta=1}^{\bar{N}} \beta K_\delta\Big[\cos\Big(\frac{2\pi\alpha\delta}{N}\Big) - 1\Big], \qquad y_{N-\alpha} = y_\alpha. \tag{A2.12}$$

Diagonalising the cyclic matrices R and B, we obtain from equation (2.12) and (A2.3) the dual parameters $\tilde{x}$ and $\tilde{y}$:

$$\tilde{x}_\alpha \equiv \exp \sum_{\delta=0}^{\bar{N}} \beta f_\delta\Big[\cos\Big(\frac{2\pi\alpha\delta}{N}\Big) - 1\Big] = \sum_{\delta=0}^{N-1} \exp(2\pi i\alpha\delta/N) y_\delta \Big/ \sum_{\delta=0}^{N-1} y_\delta$$
$$\tilde{y}_\alpha \equiv \exp \sum_{\delta=0}^{\bar{N}} \beta g_\delta\Big[\cos\Big(\frac{2\pi\alpha\delta}{N}\Big) - 1\Big] = \sum_{\delta=0}^{N-1} \exp(2\pi i\alpha\delta/N) x_\delta \Big/ \sum_{\delta=0}^{N-1} x_\delta. \tag{A2.13}$$

Since

$$x_\alpha \to D(x_\alpha) = \tilde{y}_\alpha, \qquad y_\alpha \to D(y_\alpha) = \tilde{x}_\alpha,$$

gauge and Higgs fields are exchanged by the duality transformation.

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